Prime Time Notes

Problem 1.1

Definitions:

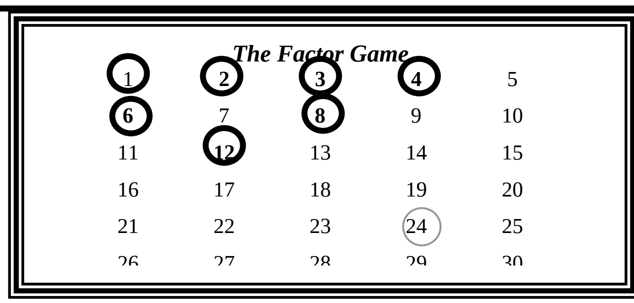
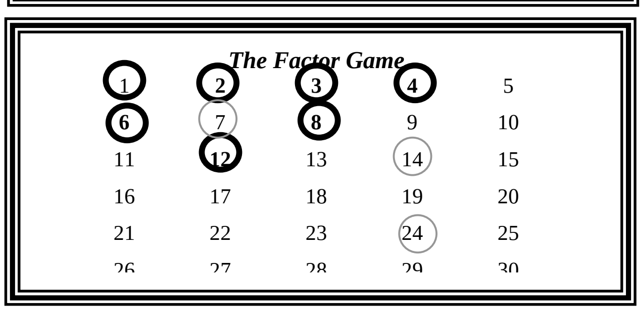
factor - one of two or more whole numbers that when multiplied together give a product
proper factor - all the factors of that number, except the number itself
abundant numbers - numbers whose sum of proper factors is greater than the number
deficient numbers - numbers whose sum of proper factors is less than the number
perfect numbers - numbers whose sum of proper factors is equal to the number

The factor game is a two-person game in which players find factors of numbers on a game board.

Rules:

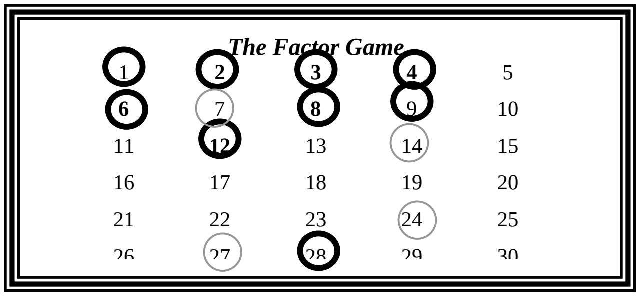
1. Player A chooses a number on the game board and circles it.
2. Using a different color, Player B circles all the proper factors of Player A's number.
3. Player B circles a new number, and Player A circles all the factors of the number that are not already circled.
4. The players take turns choosing numbers and circling factors.
5. If a player circles a number that has no factors left (that have not been circled) that player loses a turn and does not get the points for the number circled.
6. The game ends when there are no numbers remaining with un-circled factors.
7. Each player adds the numbers that are circled with his or her color. The player with the greater total is the winner.

Sample Game

Action	Game Board	Score	
		Cathy	***Keiko***
Cathy circles 24. Keiko circles 1, 2, 3, 4, 6, 8, and 12 - the proper factors of 24.		24	36
		Cathy	***Keiko***
Keiko circles 28. Cathy circles 7 and 14 - the factors of 28 that are not already circled.		24 21	36 28

Cathy circles 27.

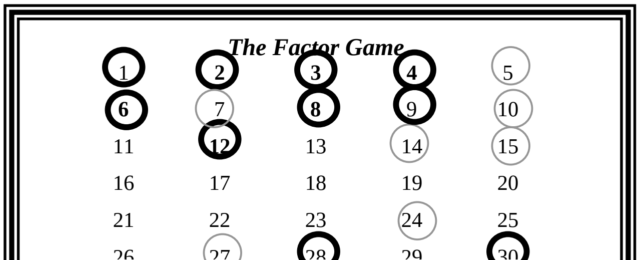
Keiko circles 9 - the only factor of 27 that is not already circled.



Cathy	Keiko
24	36
21	28
27	9

Keiko circles 30.

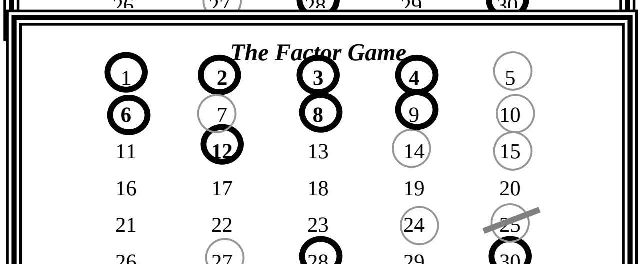
Cathy circles 5, 10, and 15 - the factors of 30 that are not already circled.



Cathy	Keiko
24	36
21	28
27	9
30	30

Cathy circles 25.

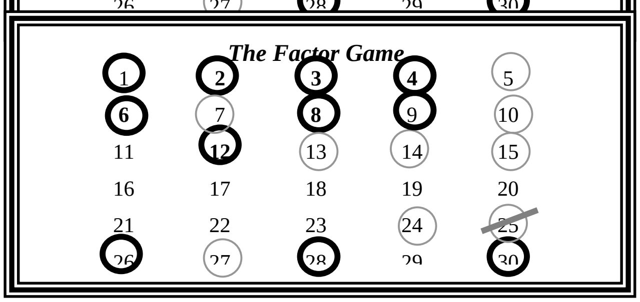
All the factors of 25 are circled. Cathy loses a turn and does not receive any points for this turn.



Cathy	Keiko
24	36
21	28
27	9
30	30

Keiko circles 26.

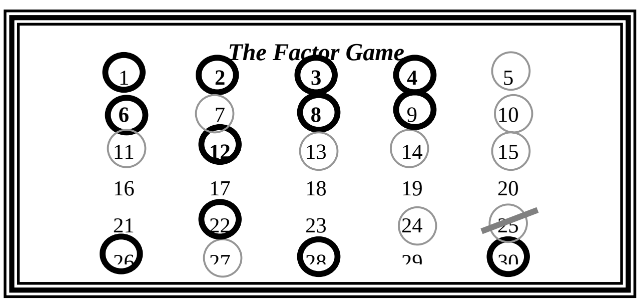
Cathy circles 13 - the only factor of 26 that is not circled.



Cathy	Keiko
24	36
21	28
27	9
30	30
13	26

Keiko circles 22.

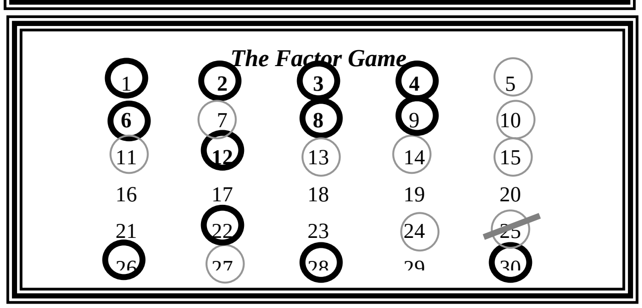
Cathy circles 11 - the only factor of 22 that is not circled.



Cathy	Keiko
24	36
21	28
27	9
30	30
13	26
11	22

No numbers remain with n-circled factors.

Keiko wins the game.



	Cathy	Keiko
	24	36
	21	28
	27	9
	30	30
	13	26
	11	22
Total	**126**	**151**

Prime Time Notes

Problem 1.2

Definitions:
prime numbers - a number with only two factors, 1 and the number itself
composite numbers - a number with more than factors of 1 and the number itself

Some numbers are better to pick than others when playing the Factor Game. To find the best numbers to pick, you can make a table of all the possible first moves. It might look like the one below.

First Move	Proper factors	My score	Opponent's score
1	none	lose a turn	0
2	1	2	1
3	1	3	1
4	1, 2	4	3
5	1	5	1
6	1, 2, 3	6	6
7	1	7	1
8	1, 2, 4	8	7
9	1, 3	9	4
10	1, 2, 5	10	8
11	1	11	1
12	1, 2, 3, 4, 6	12	16
13	1	13	1
14	1, 2, 7	14	10
15	1, 3, 5	15	9
16	1, 2, 4, 8	16	15
17	1	17	1
18	1, 2, 3, 6, 9	18	21
19	1	19	1
20	1, 2, 4, 5, 10	20	22
21	1, 3, 7	21	11
22	1, 2, 11	22	14
23	1	23	1
24	1, 2, 3, 4, 6, 8, 12	24	36
25	1, 5	25	6
26	1, 2, 13	26	16
27	1, 3, 9	27	13
28	1, 2, 4, 7, 14	28	28
29	1	29	1
30	1, 2, 3, 5, 6, 10, 15	30	42

Best first move is 29. Worst first move is 24 or 30.

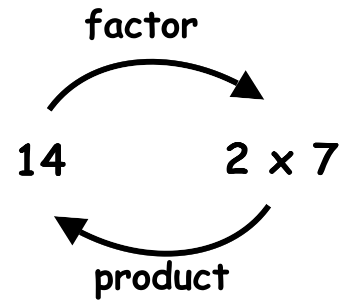
Prime Time Notes

Problem 2.1

Definitions:

multiple - the product of a given whole number and another whole number

> In the **Factor Game**, you start with a number and find its factors.
>
> In the **Product Game**, you start with factors and find their product.



The Product Game is a two-person game in which players try to get four numbers in a row (horizontally, vertically, or diagonally).

Rules:

1. Player A puts a paper clip on a number in the factor list. Player A does not marker a square on the product grid because only one factor has been marked: it takes two factors to make a product.
2. Player B puts the other paper clip on any number in the factor list (including the same number marked by Player A) then shades or covers the product of the two factors on the product grid.
3. Player A moves either one of the paper clips to another number and then shades or colors the new product.
4. Each player, in turn, moves a paper clip and marks a product. If a product is already marked, the payer does not get a mark for that turn. The winner is the first player to mark four squares in a row - up and down, across, or diagonally.

Sample Game Board

The Product Game

1	2	3	4	5	6
7	8	9	10	12	14
15	16	18	20	21	24
25	27	28	30	32	35
36	40	42	45	48	49
54	56	63	64	72	81

Factors:

1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9

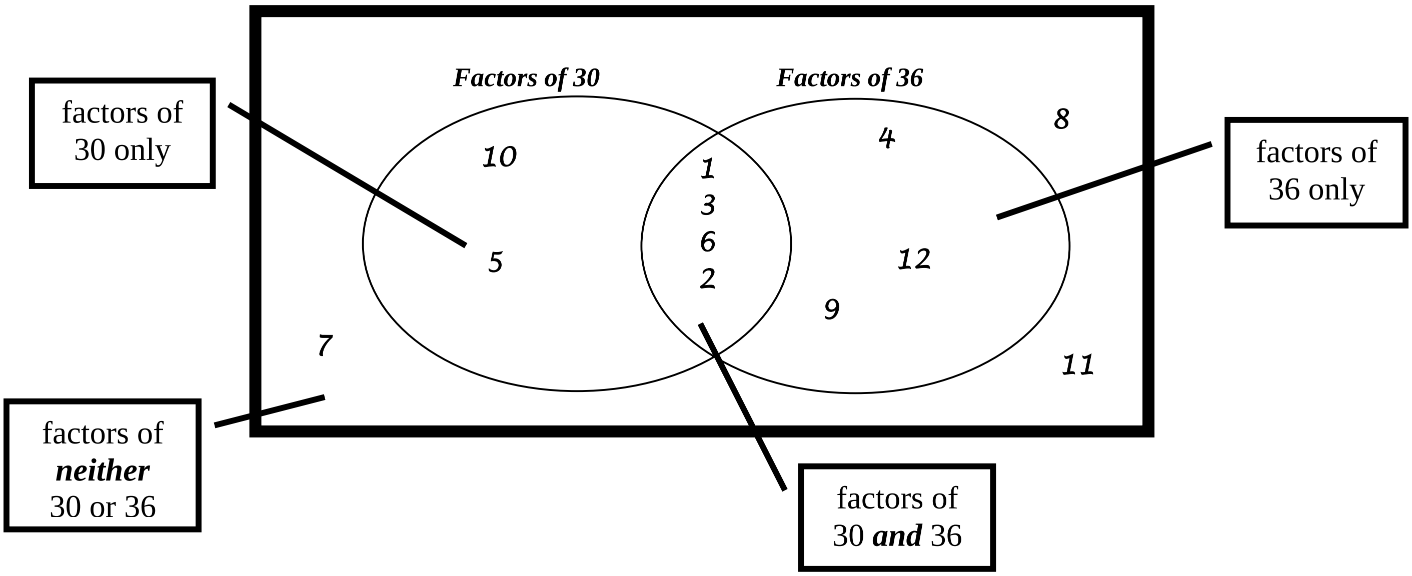
Prime Time Notes

Problem 2.3

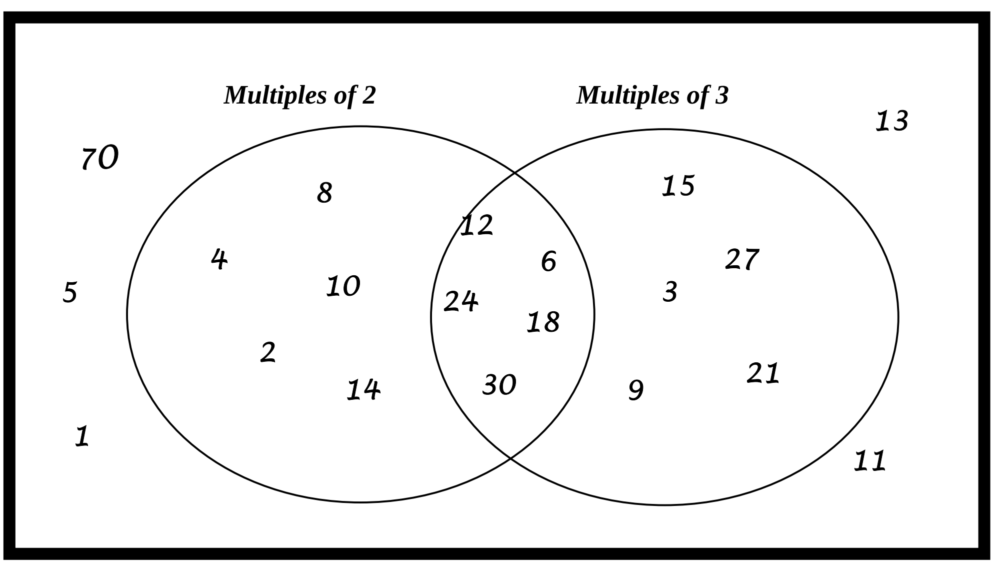
Definitions:

Venn Diagram - a diagram in which circles are used to show relationships among sets of objects that have certain attributes.

Factors and multiples are related. Sometimes it is easier to see by using a diagram. One special type of diagram that uses circles is called a Venn Diagram. Look at the following example.



Now look at some POSSIBLE answers to Problem 2.3.



Prime Time Notes

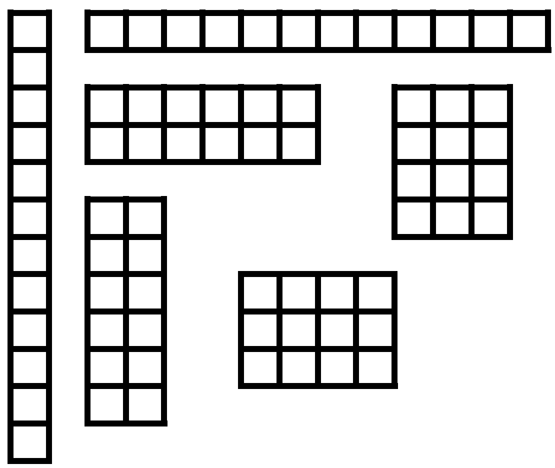
Problem 3.1

From the first two investigations you found that factors come in pairs. Once you know one factor a number, you can find another factor.

Every year, Meridian Park has an exhibit of arts and crafts. People who want to exhibit their work rent a space for $20 per square yard. All exhibit spaces must have a rectangular shape. The length and width of an exhibit space must be whole numbers of yards.

Terrapin Crafts wants to rent a space of 12 square yards.

A. Use 12 square tiles to represent the 12 square yards. Find all the possble ways the Terrapin Crafts owner can arrange the squares.



B. How are the rectangles you found and the factors of 12 related?
The factors of 12 are the dimensions of the rectangles.

Suppose Terrapin Crafts decided it wanted a space of 16 square yards.

C. Find all the possble ways that the Terrapin Crafts owner can arrange the 16 square yards.
The rectangles for 16 have dimesions of 1 x 16, 2 x 8, 4 x 4, 8 x 2 & 16 x 1.

D. How are these and the factors of 16 related?
The factors of 16 are the dimensions of the rectangles.

Prime Time Notes

Problem 3.2

Definitions:
square numbers - numbers whose tiles can be arranged to form a square

There are patterns in numbers. One way to see the patterns is to create model rectangles for a number.

Your class has made rectangles for each of the numbers 1-30. As you look at the rectangles for each number, begin to look for some patterns.

1. Which numbers have the ***most*** rectangles?

   The numbers with the most rectangles are 24 and 30.

   What kind of numbers are these?

   These numbers are compostite and abundent.

2. Which numbers have the ***fewest*** rectangles?

   The number 1 has the fewest rectangles. The numbers 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29 each have two rectangles.

   What kind of numbers are these?

   Except for the number 1, all of the numbers are prime and deficent.

3. Which numbers are square numbers?

   The square numbers, or numbers whose tiles will make a square, are 1, 4, 9, 16, and 25.

4. If you know the rectangles you can make from a number, how can you use this informatiom to list the factors of the number?

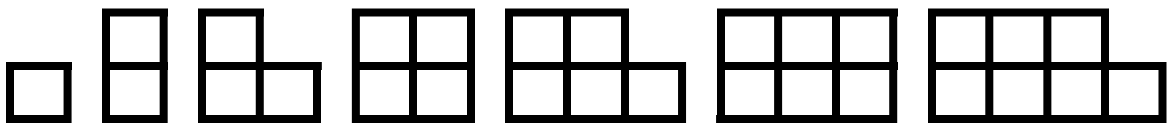
   The dimesions of the rectangles are the factors of the number.

Prime Time Notes

Problem 3.3

Definitions:
even number - a number with 2 as one of its factors
odd number - a number that does not have 2 as one of its factors

Will and his friend Jocelyn, make models for whole numbers by arranging square tiles in a special pattern. Here are their tiles models for the numbers from 1 to 7.



Make a conjecture (a reasonable guess) about whether each result below will be even or odd. Use tile models or some other method to justify your conjecture.

A. **The sum of two even numbers will be...**
The sum of two even numbers is even. If you combine the tiles, it will make a rectangle with no extra tiles.

B. **The sum of two odd numbers will be...**
The sum of two odd numbers is even. If you combine the tiles, it will make a rectangle with an extra tile.

C. **The sum of an odd number and an even number will be...**
The sum of an odd number and even number is odd. If you combine the tiles, it will make a rectangle with an extra tile.

D. **The product of two even numbers will be...**
The product of two even numbers is even. If you combine the tiles, it will make a rectangle with no extra tiles.

E. **The product of two odd numbers will be...**
The product of two odd numbers is odd. If you combine the tiles, it will make a rectangle with an extra tile.

F. **The product of an odd and an even number will be...**
The product of an even and an odd number is even. If you combine the tiles, it will make a rectangle with no extra tiles.

1. **Is zero an even number or an odd number?**
The whole numbers are placed on the number line so that every other number is odd and every other number is even. Since 1 is odd, 0 must be even.

2. **How can you tell, just by looking at the problem, whether a sum of numbers such as 127 + 8, is even or odd?**
Since 127 is odd and 38 is even, and an odd plus an even is odd, the sum of 127 and 38 must also be odd.

Prime Time Notes

Problem 4.1

Definitions:

common multiples - a multiple that two or more numbers share
common factor - a factor that two or more numbers share

Knowing about factors and multiples can help you predict what will happen in certain types of situations or cycles. Look at the multiples and factors below.

Compare the multiples of 20 and 30

multiples of 20 are:	**20,**	**40,**	**60,**	**80,**	**100,**	**120...**
multiples of 30 are:	**30,**	**60,**	**90,**	**120,**	**150,**	**180...**

Compare the factors of 12 and 30

factors of 12 are:	**1,**	**2,**	**3,**	**4,**	**6,**	**12,**		
factors of 30 are:	**1,**	**2,**	**3,**	**5,**	**6,**	**10,**	**15,**	**30**

*The numbers **60, 120, and 180** are all **common multiples** of 20 and 30.*

*The numbers **1, 2, 3, and 6** are all **common factors** of 12 and 30.*

You and your sister go to the Texas State Fair. There is the Giant Ferris wheel and the Little Ferris Wheel. If the rides begin as you as you are buckled in, find out how many seconds will pass before you and your sister are both at the bottom again.

A. **If the large wheel makes one revolution in 60 seconds and the small wheel makes one revolution in 20 seconds:**

multiples of 20: 20, 40, 60, 80, 120, ...
multiples of 60: 60, 120, 180, 240, ...

The smallest multiple that they share is 60, so they will be at the bottom after 60 seconds.

B. **If the large wheel makes one revolution in 50 seconds and the small wheel makes one revolution in 30 seconds:**

multiples of 50: 50, 100, 150, 200, ...
multiples of 30: 30, 60, 90, 120, 150, 180, ...

The smallest multiple that they share is 150, so they will be at the bottom after 150seconds.

C. **If the large wheel makes one revolution in 10 seconds and the small wheel makes one revolution in 7 seconds:**

multiples of 10: 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, ...
multiples of 7: 7, 14, 21, 28, 35, 42, 49, 56, 63, 70, 77, ...

The smallest multiple that they share is 70, so they will be at the bottom after 70 seconds

Prime Time Notes
Problem 4.2

Cicada's spend most of their lives underground. Some cicadas – commonly called 13-year locusts – come above ground every 13 years, while others – called 17-year locusts – come out every 17 years.

Stephen's grandfather told him about a terrible year when the Cicada were so numerous that they ate all the crops on his farm. Stephen conjectured that both 13-year and 17-year locusts came out that year. Assume Stephen's conjecture is correct.

A. **How many years pass between the years when both 13-year and 17-year locusts are out at the same time?**

The least common multiple is 221, so 221 years before they are out together again.

multiples of 13: 13, 26, 39, 52, 65, 78, 91, 104, 117, 130, 143, 156, 169, 182, 195, 208, <u>221</u>, …

multiples of 17: 17, 34, 51, 68, 85, 102, 119, 136, 153, 170, 187, 204, <u>221</u>, …

B. **Suppose there were 12-year, 14-year, and 16-year locusts, and they all came out this year. How many years will it be before they all come out together again?**

The least common multiple is 336, so 336 years before they are out together again.

multiples of 12: 12, 24, 36, 48, 60, 72, 84, 96, 108, 120, 132, 144, 156, 168, 180, 192, 204, 216, 228, 240, 252, 264, 276, 288, 300, 312, 324, <u>336</u>, …

multiples of 14: 14, 28, 42, 56, 70, 84, 98, 112, 126, 140, 154, 168, 182, 196, 210, 224, 238, 252, 266, 280, 294, 308, 322, <u>336</u>, …

multiples of 16: 16, 32, 48, 64, 80, 96, 112, 128, 144, 160, 176, 192, 208, 224, 240, 256, 272, 288, 304, 320, <u>336</u>, …

Prime Time Notes

Problem 4.3

Common factors and common multiples can be used to figure out how many people can share things equally.

Miriam's uncle donated 120 cans of juice and 90 packs of cheese crackers for the school picnic.

Each student is to receive the same number of cans of juice and the same number of packs of crackers.

A. **What is the largest number of students than can come to the picnic and share the food equally?**

factors of 120: 1, 2, 3, 4, 5, 6, 8, 10, 12, 15, 20, 24, 30, 40, 60, and 120

factors of 90: 1, 2, 3, 5, 6, 9, 18, 30, 45, and 90

The greatest common factor is 30, so 30 students can attend the picnic.

B. **How many cans of juice and how many packs of crackers will each student receive?**

120 cans of juice ÷ 30 students = 4 cans of juice for each student

90 packs of crackers ÷ 30 students = 3 packs of crackers for each student

If Miriam's uncle eats two packs of crackers before he sent the supplies to the school, what is the largest number of students that can come to the picnic and share the food equally? How many cans of juice and cracker packs will each student receive?

the factors of 88 are: 1, 2, 4, 8, 11, 22, 44, and 88.
The greatest common factor of 120 and 88 is 8, so 8 students can attend the picnic.

120 cans of juice ÷ 8 students = 15 cans of juice for each student
88 packs of crackers ÷ 8 students = 11 packs of crackers for each student

Prime Time Notes

Problem 5.1

Some numbers can be written as the product of several different pairs of factors. For example, 100 can be written as 1 x 100, 2 x 50, 4 x 25, 5 x 20, and 10 x 10. It is also possible to write 100 as the product of three factors, such as 2 x 2 x 25 and 2 x 5 x 10.

Using Labsheet 5.1, find as many factor strings as you can that have a product of 840. Keep a record of the strings that you find.

The Product Puzzle

30	x	14	x	8	x	7	x	210	x
x	2	x	4	x	3	x	2	x	2
105	x	2	x	5	x	84	x	56	x
x	21	x	2	x	7	x	8	x	3
40	x	20	x	4	x	7	x	5	x
x	4	x	28	x	5	x	3	x	2
6	x	8	x	21	x	2	x	105	x
x	2	x	10	x	2	x	5	x	2
32	x	3	x	14	x	60	x	56	x
x	5	x	8	x	15	x	7	x	3

Strings found the product puzzle
105 x 2 x 4

1. **Name two strings with a product of 840 that are not in the puzzle.**

   possible answers: 420 x 2 and 280 x 3

3. **If possible, name a string with a product of 840 that is longer than any string you found in the puzzle.** ***(Do not consider strings that contain 1)***

   The longest string is 2 x 2 x 2 x 3 x 5 x 7, which is in the puzzle.

4. **How do you know when you have found the longest possible string of factors for a number?**

   If all of the numbers in a string are prime, the string is the longest possible.

5. **How many distinct longest stings of factors are there for a given number? Stings are distinct if they are different in some way other than the order in which the factors are listed.** ***(Do not consider strings that contain 1)***

   Except for order, there is only one longest string for a given number.

Prime Time Notes

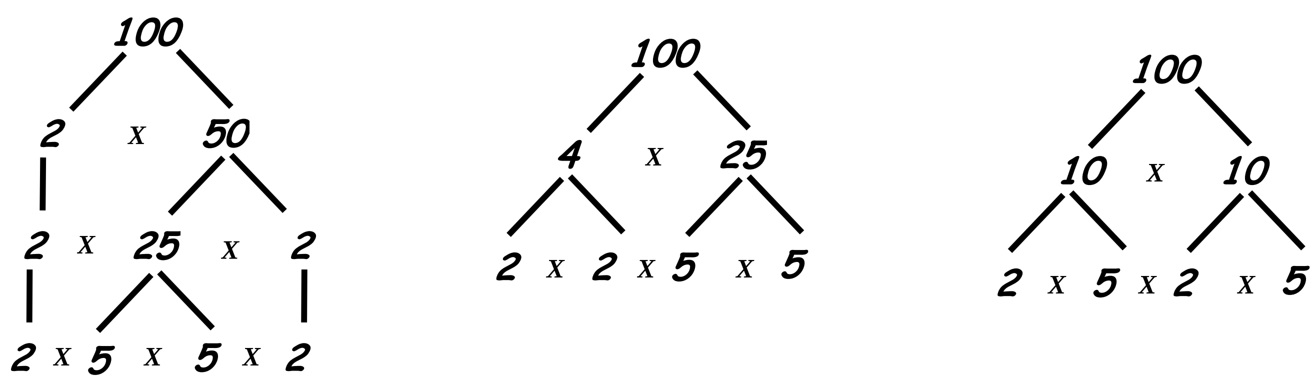
Problem 5.2

Definitions:

exponents - small raised numbers that tell you how many times a factor is repeated

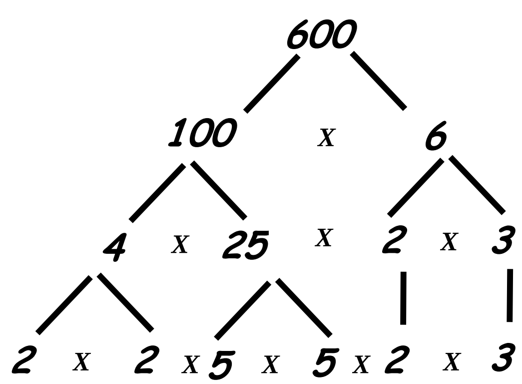
Factor Trees are used to keep track of all the prime factors in a number.

Look at each of the factor trees for the number 100.



Notice that the bottom row of each tree contains the same factors, although the order of the factors is different. All three trees indicate that the longest factorization for 100 is 2 x 2 x 5 x 5.

Try the number 600. Are there ways than shown to list the factors?



You can use a shortcut to write 2 x 2 x 5 x 5 x 2 x 3. In this shortcut, the string is written:

$$2^3 \times 3 \times 5^2.$$

The small raised numbers are exponents and tell us how many times the factor is repeated.

Prime Time Notes

Problem 5.3

Definitions:

prime factorization - a factor string for a number made up only of prime numbers

Remember:

greatest common factor - the largest factor shared by two or more numbers
least common multiple - the smallest multiple shared by two or more numbers

Heidi says that you can find the GCF and LCM of two numbers by looking at their factor strings. Look at the factor strings for 24 and 60.

24 = 2 x 2 x 2 x 3   60 = 2 x 2 x 3 x 5

Heidi claims that the GCF of two numbers is the product of the longest string of prime factors that the numbers have in common.

24 = 2 x <u>*2 x 2 x 3*</u>   60 = <u>*2 x 2 x 3*</u> x 5

According to Heidi's method, the GCF of 24 and 60 is 12, since 2 x 2 x 3 =12.

Heidi also claims that the LCM of two numbers is the product of the shortest string that contains the prime factorization of both numbers.

The prime factorization of 24   *The prime factorization of 60*

<u>*2 x 2 x 2 x 3*</u> x 5   2 x <u>*2 x 2 x 3 x 5*</u>

According to Heidi's method, the LCM of 24 and 60 is 120, since 2 x 2 x 2 x 3 x 5 = 120.

Find the GCF and LCM of 48 and 72.

48 = 2 x 2 x 2 x 2 x 3   72 = 2 x 2 x 2 x 3 x 3

GCF = 2 x 2 x 2 x 3 = 24   LCM = 2 x 2 x 2 x 2 x 3 x 3 = 144

Find the GCF and LCM of 30 and 54.

30 = 2 x 3 x 5   54 = 2 x 3 x 3 x 3

GCF = 2 x 6 = 6   LCM = 2 x 3 x 3 x 3 x 5 = 270

Prime Time Notes

Problem 6.1

There are 1000 lockers in the long hall of Westfalls High. Before the first day of school, the custodian closes all of the locker doors.

☞ On the first day of school Mrs. Millie Meter, a crazy math teacher, has a student run down the hall and open every locker door.

☞ A student right behind him closes the doors of lockers 2, 4, 6, 8, and so on to the end of the line.

☞ A third student changes the state of the locker doors *(if the door was open, he closed it and if the door was closed, he opened it)* on lockers 3, 6, 9, 12, and so on to the end of the locker.

☞ A fourth student changes the state of the doors for lockers 4, 8, 12, 16, and so on to the end of the line.

☞ Student 5 changes the state of every fifth door and student 6 changes the state of every sixth door, and so on until all 1000 students have had a turn.

When the students are finished, which locker doors are open?

The doors on the lockers with square numbers will be open.
(1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, 324, 361, 400, 441, 484, 529, 576, 625, 676, 729, 784, 841, 900, and 961)

1. Some of the lockers that touched by exactly...
   two students - 2, 3, 4, 7, 11, 17, 17, 19, 23,...
   three students - 4, 9, 25, 49, 121, 169, 289, 361, 529, 841, 961
   four students - 6, 8, 10, 15, 27, 35,...
2. The first locker touched by both student 6 and 8 was locker number 24.
3. Students 1, 2, 3, 4, 6, and 12 touched lockers 24 and 36.
4. Students 1, 2, 4, 5, 10, and 20 touched lockers 100 and 120.
5. The first student to touch both lockers 100 and 120 was student 600.